ISSN 2229-5518

Measurement of Lipschitz Exponent (LE) using Wavelet Transform Modulus Maxima (WTMM)

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Abstract: Singularity and dynamical behavior are two important aspects in signal processing that carries most of signal information. A remarkable property of the wavelet transform is its ability to characterize the local regularity of functions. In mathematics, this local regularity is often measured with Lipschitz exponents (LE). The singularity, by means of a Lipschitz exponent of a function, is measured by taking a slope of a log-log plot of scales and wavelet coefficients along modulus maxima lines of a wavelet transform [1]. At present, most of the existing methods of measuring LE using wavelet transform are derived from the previous work of Mallat and Hwang in [1], which equals LE to the maximum slope of straight lines that remain above the wavelet transform modulus maxima (WTMM) curve in the log-log plot of scale s versus WTMM. However this method is not always robust and precise especially in noise environment, because it is only the particular case of the equation (25) in [1]. In this paper we present the measurements of lipschitz exponent using wavelet transform with a new area based objective function. The results of experiment demonstrate that this method is more precise and robust.

Key words: Lipschitz exponent, Wavelet Transform, local regularity, singularity, slope

1 Introduction

Singularities and irregular structures often carry the most important information in signals. Because singularity often carries the most important information contained in a signal, singularity analysis has emerged as a multiple-area problem solving method in recent years [2], [3], [4], [5], [6] and [7]. In mathematics, the singularity is usually measured with Lipschitz exponent (LE). It is a real number that can characterize the local regularity or smoothness in a signal. The definition of LE is given in [1]. The signal singularity refers to the intermittent points or discontinuous derivative of the signal. In mathematics, the sharpness of an edge can be described with Lipschitz Exponent. Local lipschitz can be efficiently measured by wavelet transform. The relationship between the modulus of wavelet transform and lipschitz exponent can be described as theorem 1.The WTMM representation of a signal records the values and locations of local maxima of its wavelet transform modulus. They proved that the local lipschitz exponent of a signal can be estimated by tracing the evaluation of its WTMM across scales. From the estimated lipschitz exponent and with some other a priori information of the signal, an effective denoiseing method can be developed. Although the WTMM based algorithms give a promising performance in many aspects, the irregular sampling nature of the WTMM complicates the reconstruction process. This paper is organized as follows. The wavelet transform and a tutorial review on lipschitz exponent are briefly introduced in section II. The Lipschitz exponent measuring with WTMM is presented in section III. In section IV, we present the experiment procedure to measure the LE from WTMM and result analysis is presented in section V. Finally, section VI gives some concluding remarks.

2. Fundamental Concepts:

This second section reviews the main properties of the wavelet transform and lipschitz exponent. The formalism of the continuous wavelet transform was first introduced by Morlet and Grossman [8]. Let ψ (t) be a complex valued function. The function ψ (t) is said to be a wavelet if and only if its Fourier transform $\hat{\psi}(\omega)$ satisfies

$$\int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega = \int_{-\infty}^{0} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega = C_\psi < +\infty \quad (1)$$

This condition implies that

$$\int_{-\infty}^{+\infty} \psi(t)\, dt = 0 \quad (2)$$

The continuous wavelet transform of a function f(t) ∈ $L^2(R)$ with respect to the wavelet ψ (t) is defined as

$$Wf(u,s) \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) dt \quad (3)$$

where ψ* denotes the complex conjugate of ψ.

A wavelet ψ (t) is said to have n vanishing moments if and only if for all positive integers k < n, it satisfies,

$$\int_{-\infty}^{+\infty} t^k \psi(t)\, dt = 0 \quad (4)$$

A popular wavelet in practice is the nth derivation of the Gaussian function

$$\psi_n(x) = -\frac{d^n}{dx^n} e^{\frac{-x^2}{2}} \quad (5)$$

When performing wavelet singularity analysis, the number of vanishing moments is very important, as it provides an upper bound measurement for singularity characterization. Lipschitz exponent is a measurement of the strength of a singularity. Mallat and Hwang [1] showed that the LE can be computed by WTMM of signals.

Singular exponent: A function f(x) is said to be lipschitz α, for $0 \le \alpha \le 1$, at a point x_0, if and only if there exists a constant A such that for all points x in a neighborhood of x_0

$$|f(x) - f(x_0)| \le A|x - x_0|^\alpha \quad (6)$$

The function f(x) is uniformly lipschitz α for any $x_0 \in (a,b)$ and $x \in (a, b)$. We say that f(x) is singular in x_0 if it is not Lipschitz 1 in x_0. If a function is Lipschitzα , for $\alpha > 0$, then it is continuous in x_0. If f(x) is discontinuous in x_0 and bounded in a neighborhood of x_0, then it is lipschitz 0 in x_0. If f(x) is continuously differentiable then it is lipschitz 1 and thus not singular.
We suppose that the ψ (t) has a compact support, is n times continuously differentiable and is the nth derivatives of a smoothing function. The theorem 4 of [1] can be rewritten as:
Theorem 1:

Let f(x) be a tempered distribution whose wavelet transform is well defined over (a, b), and let $x_0 \in (a, b)$. We suppose that there exists a scale $s_0 > 0$, and a constant C, such that for $x \in (a, b)$ and $s < s_0$, all the modulus maxima of Wf(s, x) belong to a cone defined by

$$|x - x_0| \leq Cs \qquad (7)$$

Then, at all points $x_1 \in (a, b)$, $x_1 \neq x_0$,f(x) is uniformly Lipschitz n in a neighborhood of x_1. Let
$\alpha < n$ be a non-integer. The function f(x) is lipschitz α at x_0 , if and only is there exists a constant A such that at each modulus maxima (s, x) belong to a cone defined by (7)

$$|Wf(s,x)| \leq As^{\alpha} \qquad (8)$$

By substituting S_i and S_{i+1} into equation (9), throughout simple derivation, lipschitz exponents can be expressed in the following form

$$\alpha = \frac{log_2\left|\frac{Wf(s_{(i+1)},x)}{Wf(s_i,x)}\right|}{log_2\left|\frac{s_{(i+1)}}{s_i}\right|} \qquad (9)$$

The value of lipschitz exponent α can be reflects the degree of failure. The smaller the lipschitz exponent is the stronger the curve deviates.

(9)

3. LE Measuring with WTMM:

Based on theorem 1, there are some existing methods used to estimate LE [1][4][12][14]. Equations (6) and (8) imply that $|Wf(s, x)| \leq O(s^{\alpha})$ inside a cone $|x-x_0| \leq Cs$ [1], where C is the support of the mother wavelet. This cone is the so-called "cone of influence" (COI), as shown in Fig.1. Mallat and Hwang furthered [1, Th.4] and proposed to estimate the lipschitz exponent of a singularity by tracing its WTMM curves across scales inside the COI. They showed that the local regularity of certain types of non-isolated singularities in the signal can be characterized by using the WTMM. They also showed that the decay of the expected WTMM value of a wide noise across scales is proportional to $1/2^j$, where $s=2^j$ .This means that the WTMM curve of noise are expected to decay across scales at least at a rate of $1/2^j$ or even not propagate to coarser scales .This is not the case for regular signals and edges .Since signals edges possess zero lipschitz exponents and regular signals possess positive lipschitz exponent , the corresponding WTMM will be the same, if it does not increase , when scale increase. Equ(8) is equivalent to

$$log_2|Wf(s,x)| \leq log_2(A) + \alpha log_2(s) \qquad (10)$$

$$f(A,\alpha) = \left|(log_2(A) + \alpha log_2(s_{small}) - log_2|Wf(u, s_{small}|)\right| + \left|(log_2(A \right. \qquad (11)$$

If the wavelet transform maxima satisfy the cone distribution imposed by theorem 4,in[1],(10) proves that the lipschitz regularity at x_0 ,is the maximum slope of straight lines that remain above log |Wf(s, x)|,on a logarithmic scale. The fact that all modulus maxima remain in a cone that points to x_0 also implies that f(x) is lipschitz n at all points $x \in]a, b[$, $x \neq x_0 x$)

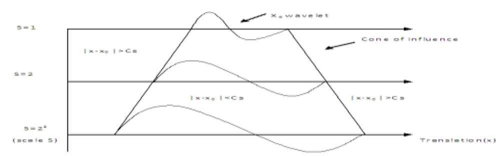

Figure. 1. "cone of influence"(s, x) plane of the wavelet function (C)in different scales.

3. Lipschitz exponent (α) measurement procedure:

From the Theorem.1, we can measure the Lipschitz exponent using the following algorithm:

1. Compute straight line $l(log_2(s))$ connecting ($log_2(s_{small})$, $log_2|Wf(u ,s_{small})|$) and ($log_2(s_{max})$, $log_2|Wf(u, s_{max})|$). If $l(log_2(s)) \geq log_2|Wf(u, s)|$, return the intercept $log_2(A)$ and slope α of $l(log_2(s))$, go to 7),otherwise, go to 2).
2. Let $s=s_{max}$ and $f(A,\alpha) = C$, where C is a constant large enough.
3. Compute tangent $l(log_2(s))$ at ($log_2(s)$, $log_2|Wf(u,s)|$). If $l(log_2(s)) \geq log_2|Wf(u,s)|$, go to 4). Otherwise go to 6).
4. Compute (11), record the result f of (11) and the intercept $log_2(A)$ and slope α of $l(log_2(s))$. If $f < f(A,\alpha)$, $f(A,\alpha) = f$ and LE = α.
5. If $s = s_{min}$, go to 7). Otherwise go to 6).
6. $s = s - \Delta log_2(s)$, go to 3
7. Output LE = α.

4. Result Analysis:

Theorem 1 proves that the wavelet transform is particularly well adopted to estimate the local regularity of function. When a function is approximated at a finite resolution, strictly speaking, it is not meaningful to speak about singularities, discontinuities and Lipschitz exponents. This is illustrated by the fact that we cannot compute the asymptotic decay of the wavelet transform amplitude at scales smaller than one. In this work we used the function f(x) shown in Fig.2 will be used for testing the capabilities of the wavelet to determine the regularity.

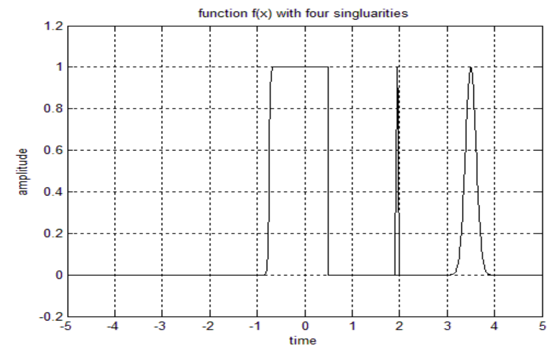

Figure.2. Function f(x) with four singularities at abscissa 413, 551, 696, and 802.

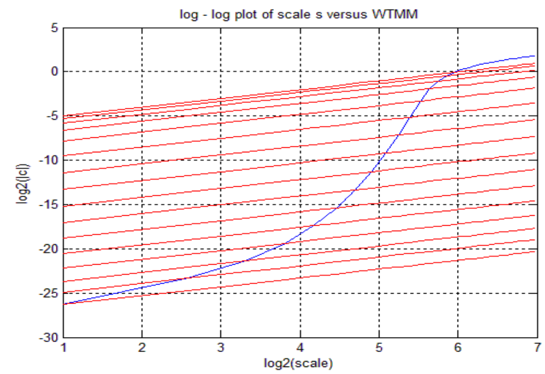

Figure 4.log - log plot of scale s versus WTMM with tangent line

Continuous Wavelet transform of function f(x) shown in Fig.3. In Fig.3, the discontinuity appears clearly from the fact that |Wf(s, x)| remains approximately constant over a large range of scales, in the neighborhood of the abscissa 551. A negative lipschitz exponent corresponds to sharp irregularities where the wavelet transform modulus increases at fine scales. At the abscissa 696, the signal of Fig.2 has such a discrete Dirac. The wavelet transform maxima increase proportionally to s^{-1}, over a large range of scales in the corresponding neighborhood.

The log-log plot of scale s versus WTMM shown in Fig.4, then to find the slope of corresponding scale and coefficient line using lipschitz exponent (α) measurement procedure. We determined lipschitz exponent function (α) and compared refer Table.1. LE with objective function is more accurate value. Because we use the appropriate known edge of α, algorithm searches the optimal result along $\log_2|Wf(u,s)|$ curve only, and the problem of initialization of A and α can be avoided. The adopted wavelet $\psi(x)$ is the second derivative of a Gaussian function. We denote S_{small}=2 S_{max}=64 and Δlog2s=0.0326, and for this method we adopt the initial function values A=2 and α=1.

TABLE 1
COMPARISON OF LE (A) WITH HAWANG AND OBJECTIVE FUNCTION.

S. NO	SINGULARITY AT ABSCISSA	LE IN [1] HAWANG	LE IN OBJECTIVE FUNCTION
1	413(-0.92)	2.4497	1.1648
2	551(0.5)	0	0.0047
3	692(2)	-0.1669	-0.0318
4	802(3.5)	2.3635	0.6513

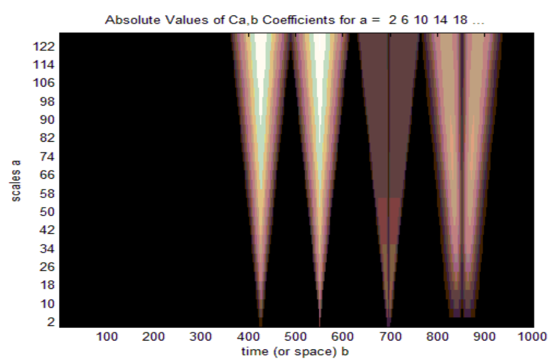

Figure 3. Continuous Wavelet transform of function f(x)

5. Conclusion:

We proved that the wavelet transform modulus maxima detect all the singularities of a function and we described strategies to measure their Lipschitz regularity. This mathematical study provides algorithm for characterizing singularities of irregular signals such as the multiracial structures observed in physics. In this paper, we have presented a novel method for measuring Lipschitz exponent. We use the area between the straight line satisfying (10) and the curve of WTMM in a finite scale interval in the log-log plot of scale s versus WTMM as the objective function. The exponent results demonstrated out algorithm had better precision and robustness.

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